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Some Remarks on a Model of Supersymmetric Quantum Mechanics

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We develop a recently proposed model within supersymmetric quantum mechanics that puts a group structure on the creation and annihilation operators. We apply the scheme to a variety of quantum mechanical problems and work out a two-term energy recursion equation when the overall group structure is $U(1, 1)$.

1. INTRODUCTION

The problem of the factorizability (Lahiri *et al.*, 1990; Roy *et al.*, 1991) of a Schrödinger Hamiltonian and the existence of supersymmetry (SUSY) in quantum mechanical systems are related. Indeed, if we are able to factorize a Hamiltonian for a given potential, we can immediately construct a pair of Hamiltonians (which are supersymmetric partners really) whose energy levels are in one to one correspondence. The principle of SUSY relates the characteristics of a boson with a fermion—it was only in 1981 that Witten (1981) showed the relevance of SUSY in quantum mechanics.

To develop a supersymmetric theory, one has to define a set of supercharges Q_i which obey the graded algebra

$$\{Q_i, Q_j\} = H_s, \qquad [Q_i, H_s] = 0 \tag{1}$$

where $i, j = 1, \ldots, N$ and H_s is the governing Hamiltonian.

For $N = 2$ the linear combinations $Q = (Q_1 + iQ_2)/2$ and $Q^+ = (Q_1 - iQ_2)/2$ transform the algebra (1) into

$$[Q, Q^+] = H_s, \qquad Q^2 = (Q^+)^2 = [Q, H_s] = [Q^+, H_s[= 0 \tag{2}$$

We refer to (2) as the basic structure for the $N = 2$ supersymmetric quantum mechanics.

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To realize (2) in physical systems, we have to introduce a pair of linear differential operators (A, A^+) in terms of which Q and Q^+ read

$$Q=\begin{pmatrix}0 & A\\ 0 & 0\end{pmatrix}, \qquad Q^+=\begin{pmatrix}0 & 0\\ A^+ & 0\end{pmatrix} \tag{3}$$

The above choice brings out the nilpotent character of Q and Q^+: $Q^2=(Q^+)^2=0$.

Given (3), the supersymmetric Hamiltonian H_s may be expressed in a matrix form

$$H_s=\begin{pmatrix}AA^+ & 0\\ 0 & A^+A\end{pmatrix}=\begin{pmatrix}H_- & 0\\ 0 & H^+\end{pmatrix} \tag{4}$$

The eigenfunctions of H_+ and H_- satisfy

$$H_+|n\rangle=E_n|n\rangle, \qquad H_-|n-1\rangle=E_n|n-1\rangle \tag{5}$$

If the ground state $|0\rangle$ $(n=0)$ is nondegenerate, then SUSY is said to be unbroken. One may chose $|0\rangle$ to be associated with H_+. For higher values of n we get a supersymmetric spectrum, the eigenvalues of which are paired.

The simple choice

$$A=\left(\frac{d}{dx}+W\right), \qquad A^+=\left(\frac{-d}{dx}+W\right) \tag{6}$$

leads to a one-dimensional supersymmetric formulation which consists of the following pair of Hamiltonians:

$$\begin{aligned}H_+&=-\partial_x^2+W^2-W'\\ H_-&=-\partial_x^2+W^2+W'\end{aligned} \tag{7}$$

where the prime denotes derivatives with respect to x. The function $W(x)$ in (6) is referred to as the superpotential.

The supersymmetric Hamiltonian H_s is connected to the Schrödinger Hamiltonian through the solution of a nonlinear equation of the Riccati type, $V-E_0=\frac{1}{2}(W^2-W')$, where E_0 is the lowest energy level.

Let us further note that the operators A and A^+ obey

$$\{A, A^+\}=-\frac{d^2}{dx^2}+W^2 \tag{8a}$$

$$[A, A^+]=\frac{dw}{dx} \tag{8b}$$

Actually, as pointed out by Jannussis *et al.* (1990), the set of relations (6)–(8) constitute a particular case of the Lie-admissible aspect of Santilli's (1979) theory.

Recently, Lahiri *et al.* (LRB) (1988) considered the interesting possibility of imposing a four-parameter group structure on the raising and lowering operators A and A^+. To this end, the following representations were chosen:

$$\begin{aligned} A &= \exp(iy)[k(x)\,\partial x - ik'(x)\,\partial y + u(x)] \\ A^+ &= \exp(-iy)[-k(x)\,\partial x - ik'(x)\,\partial y + u(x)] \end{aligned} \tag{9}$$

where $k(x)$ and $u(x)$ are arbitrary functions of x. Note that the presence of an additional parameter y in (9) facilitates the closing of A and A^+ with

$$A_3 = -i\,\partial y, \qquad [A, A^+] = -2aA_3 - bI, \qquad [A_3, A] = A, \qquad [A_3, A^+] = -A^+ \tag{10}$$

In (10) the functions a and b are related to $k(x)$ and $u(x)$ through

$$\begin{aligned} a &= k'^2(x) - K(x)k''(x) \\ b &= 2[k'(x)u(x) - k(x)u'(x)] \end{aligned} \tag{11}$$

LRB emphasized that y is to be looked upon as an auxiliary parameter and not to be confused with an extra spatial dimension. This means that for a physical eigenvalue problem the square of the modulus of the eigenfunction must be independent of y.

In the spirit of equation (8a) the Hamiltonian in their scheme was proposed to be (Lahiri *et al.*, 1988)

$$H = -\tfrac{1}{2}\{A, A^+\} = -K^2\,\partial^2 x + (u - ik'\,\partial y)^2 - ik'u\,\partial y \tag{12}$$

The partner Hamiltonians H_+ and H_- in this version of supersymmetric theory read

$$\begin{aligned} H_+ &= A^+A \\ &= -K^2\,\partial x^2 + ikk''\,\partial y - ku' + k'u + (u - ik'\,\partial y)^2 \\ &\quad + uk\,\partial x - ik'^2\,\partial y - ik'u\,\partial y \\ H_- &= AA^+ \\ &= -K^2\,\partial x^2 - ikk''\,\partial y + ku' - k'u + (u - ik'\,\partial y)^2 \\ &\quad - uk\,\partial x + ik'^2\,\partial y - ik'u\,\partial y \end{aligned} \tag{13}$$

It may be remarked that the above scenario reduces to Witten's scheme by going over to $a = 0$ ($k = 1$), $b = -W'$, and $u = \frac{1}{2}W$ and ignoring the auxiliary parameter.

The model of LRB has been looked into by Jannussis *et al.* (1990), who have studied the Hamiltonian H in (12) to derive a two-term recursion equation involving the energy eigenvalue (we shall return to this point later). More recently, Chuan (1990) has proposed a set of coupled equations, a particular class of which yields the scheme of LRB. The purpose of this work is to pursue this scheme in greater detail and show that for various choices of $u(x)$, one can establish contact with a variety of quantum mechanical systems, such as the box problem, the Morse and Coulomb potentials, and also the isotropic oscillator. The possibility of linking (12) to the infinite-deep-well problem has already been considered by Lahiri *et al.* (1988); it corresponds to the simplest case when $u(x) = 0$. In this paper we also show that owing to errors present in some of the equations in Lahiri *et al.* (1988), the solutions of Jannussis *et al.* (1990) call for a reinvestigation. This we shall take up in the concluding portions of the paper.

2. PARTICLE IN A BOX

Setting $u(x) = \tan x$ and $k(x) = 1$ in the scheme (13), we get

$$\begin{aligned} A &= \exp(iy)\,(\partial x + \tan x) \\ A^{+} &= \exp(-iy)\,(-\partial x + \tan x) \end{aligned} \tag{14}$$

The above pair give

$$H_{+} = A^{+}A = -\partial_x^2 - 1 \tag{15a}$$

for $-\infty < W(x) < \infty$, $-\pi|2 < x < \pi|2$, along with the supersymmetric partner

$$H_{-} = AA^{+} = -\partial^2 x + (\sec^2 x + \tan^2 x) \tag{15b}$$

It should be remarked that SUSY relates the spectra of H_{+} and H_{-} through $E_{-}^{(n)} = E_{+}^{(n+1)}$, $n = 0, 1, 2, \ldots$. Further, the eigenfunctions and eigenvalues of H_{+} are given by (Filho, 1990)

$$\begin{aligned} \psi_{+,n}(x) &= \begin{cases} (2/\pi)^{1/2} \sin nx; & n \text{ even} \\ (2/\pi)^{1/2} \cos nx; & n \text{ odd} \end{cases} \\ E_{+}^{(n)} &= n^2 - 1; \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{16}$$

One can verify that the above steps constitute supersymmetrization of the "particle in a box" problem. With H_{+} given by (15a), the Schrödinger potential corresponds to $v(x) = 0$ for $|x| < \pi/2$ and $v(x) = \infty$ for $x = \pi/2$.

The eigenvalue spectrum is $E^{(n)} = n^2$, $n = 1, 2, \ldots$, and the ground state has a $\cos x$ form.

A few words about the new potential generated in (13): As emphasized by Sukumar (1985), one can construct a hierarchy of potentials by repeatedly factorizing the Hamiltonian once we get to (15a). Indeed in this way the adjacent members of the hierarchy do turn out to supersymmetric partners. Thus for the system (15b) a series of $\sec^2 x$ potentials is generated whose spectrum can be readily calculated.

3. MORSE POTENTIAL

Fixing $K = 1$ is consistent with $a = 0$. Setting

$$u = \frac{ex}{n} + (\tfrac{1}{2} - n) \tag{17}$$

we arrive at

$$\begin{aligned} A &= \exp(iy)\left\{\partial x + \left[\frac{ex}{n} + (\tfrac{1}{2} - n)\right]\right\} \\ A^{+} &= \exp(-iy)\left\{-\partial x + \left[\frac{ex}{n} + (\tfrac{1}{2} - n)\right]\right\} \end{aligned} \tag{18}$$

So the partner Hamiltonians H_+ and H_- in the supersymmetric theory are

$$H_+ = A^+A = -\partial_x^2 + \frac{e^{2x}}{n^2} - 2e^x + (\tfrac{1}{2} - n)^2 \tag{19a}$$

$$H_- = AA^+ = -\partial_x^2 + \frac{e^{2x}}{n^2} + \frac{2e^x}{n}(1 - n) + (\tfrac{1}{2} - n)^2 \tag{19b}$$

On the other hand, using the form (12), the Hamiltonian becomes

$$\left\{-\partial_x^2 + \left[\frac{e^{2x}}{n^2} + \frac{2}{n}e^x(1 - n) + (\tfrac{1}{2} - n)^2\right]\right\}\phi_n = E_n\phi_n \tag{20}$$

We are thus led to a Morse potential whose partners are given by the components in (19a) and (19b). Typically the Morse potential is

$$V = K\exp\left(\frac{2x}{a}\right) - 2k\exp\left(\frac{x}{a}\right) \tag{21}$$

whose spectrum is (Lahiri *et al.*, 1990; Sukumar, 1985)

$$E^{(n)} = \frac{-\alpha^2}{2a^2} + \frac{\alpha(n + \frac{1}{2})}{a^2} - \frac{2(n + \frac{1}{2})^2}{a^2} \tag{22a}$$

where $n = 0, 1, \ldots, N - 1$ and support N bound states. Note that in the above $\alpha = a(2k)^{1/2}$ and N is the largest integer less than $(\alpha + \frac{1}{2})$. The ground-state wave function is

$$\psi^0(x) \sim \exp\left[-(\alpha - \tfrac{1}{2})\frac{x}{a} - \alpha\exp\left(-\frac{x}{a}\right)\right] \tag{22b}$$

As in the "particle in a box" problem, here also one can construct a chain of potentials from (18). The $(N + 1)$-member hierarchy is found to correspond to a set of Morse potentials.

4. COULOMB PROBLEM

The radial equation of the Coulomb problem is (Lahiri *et al.*, 1990)

$$\left(-\frac{d^2}{dr^2}-2E_n-\frac{2}{r}+\frac{l(l+1)}{r^2}\right)\chi_{nl}=0 \tag{23}$$

where n and l are the principal and angular momentum quantum numbers, respectively.

Further, $\chi_{nl}(0)=0$, $E_n=-1/(2n^2)$, $r=\frac{1}{2}mZe^2x$, Z is the nuclear charge, and $r\in(0,x)$ is the radial coordinate.

Making the transformation $x=\ln r$ and redefining $\chi_{nl}(r)=e^{x|2|}\phi(x)$ brings (23) to the following form:

$$\left[-\frac{d^2}{dx^2}-2E_n\,e^{2x}-2\,e^x+(l+\tfrac{1}{2})^2\right]\phi(x)=0 \tag{24}$$

Incidentally, the Hamiltonian of the system (24) coincides with the form in (19a), which in turn is consistent with the choice of u made in (17).

From (24), we identify the superpotential $W(x)$ to be

$$W(x)=\frac{e^x}{n}+(\tfrac{1}{2}-n) \tag{25}$$

It leads to the pair

$$V_+=e^{2x}/n^2-2e^x+(\tfrac{1}{2}-n)^2 \tag{26a}$$

$$V_-=e^{2x}/n^2-2(1-1/n)\,e^{2x}+(\tfrac{1}{2}-n)^2 \tag{26b}$$

Transforming back to the variable r, the form (26b) gives the supersymmetric partner to (23),

$$\left[-\frac{d^2}{dr^2}+\frac{1}{n^2}-2\left(1-\frac{1}{n}\right)\frac{1}{r}+\frac{l(l+1)}{r^2}\right]\chi_{nl}(r)=0 \tag{27}$$

Interpreting $(1-1/n)r$ as the running variable by dividing (27) by $(1-1/n)^2$, we see (Haymaker and Rau, 1986; see also Kostelecky and Nieto, 1985) that (27) describes the state's nuclear charge $Z(1-1/n)$. Thus the degeneracy arises between states of the same l but different n and Z.

5. ISOTROPIC OSCILLATOR

Setting $K=1$ and $u(x)=[e^x/2-(n+\frac{1}{2})/2]$ in the representation (9) yields

$$\begin{aligned} A&=\exp(iy)\left[\partial_x+\left(\frac{e^x}{2}-\frac{n+\frac{1}{2}}{2}\right)\right]\\ A^+&=\exp(-iy)\left[-\partial_x+\left(\frac{e^x}{2}-\frac{n+\frac{1}{2}}{2}\right)\right]\end{aligned} \tag{28}$$

These give

$$H_+ = -\partial_x^2 + \frac{e^{2x}}{4} - \tfrac{1}{2}(n+\tfrac{3}{2})e^x + \tfrac{1}{4}(n+\tfrac{1}{2})^2 \tag{29a}$$

$$H_- = -\partial_x^2 + \frac{e^{2x}}{4} - \tfrac{1}{2}(n-\tfrac{1}{2})e^x + \tfrac{1}{4}(n+\tfrac{1}{2})^2 \tag{29b}$$

Further, from equation (12) we deduce

$$-\frac{\partial^2 \phi_n}{\partial x^2} + \left[\frac{e^{2x}}{4} - \tfrac{1}{2}e^x(n+\tfrac{1}{2}) + \tfrac{1}{4}(n+\tfrac{1}{2})^2\right]\phi_n = E_n \phi_n \tag{29c}$$

Now the radial equation for the isotropic oscillator is

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \tfrac{1}{2}r^2 + \frac{l(l+1)}{r^2} - E_n\right]\chi_{nl}(r) = 0 \tag{30}$$

where $n = 1+2, \ldots$. Transforming (30) to the full line by the change of variable $x = 2\ln r$ and putting $\chi_{nl}(r) = e^{x/4}\phi(x)$ gives

$$\left[-\frac{d^2}{dx^2} + \tfrac{1}{4}e^{2x} - \tfrac{1}{2}E_n e^x + \tfrac{1}{4}(l+\tfrac{1}{2})^2\right]\phi(x) = 0 \tag{31}$$

We see that the Hamiltonian implied by (19) exactly coincides with (29a). Certainly the supersymmetric system (29) is a particular case of the model (12) and (13).

From (31) the superpotential may be identified to be (Lahiri *et al.*, 1990) $W(x) = e^x - \tfrac{1}{2}(n+\tfrac{1}{2})$, leading to

$$\begin{aligned} V_+ &= \tfrac{1}{4}e^{2x} - \tfrac{1}{2}(n+\tfrac{3}{2})e^x + \tfrac{1}{4}(n+\tfrac{1}{2})^2 \\ V_- &= \tfrac{1}{4}e^{2x} - \tfrac{1}{2}(n-\tfrac{1}{2})e^x + \tfrac{1}{4}(n+\tfrac{1}{2})^2 \end{aligned} \tag{32}$$

Taking V_- to the half-line, we find that the partner equation turns out to be

$$\left[-\frac{d^2}{dr^2} + r^2 + \frac{l(l+1)}{r^2} + 2(2-E_n)\right]\chi_{nl}(r) = 0 \tag{33}$$

We thus see that SUSY gives rise to degeneracies (Haymaker and Rau, 1986; Kostelecky and Nieto, 1985) with the energy difference between equations (30) and (33) being a factor of 2 units as in $n = 1, 1+2, 1+4, \ldots$.

Finally, we examine the eigenvalue equation

$$Hg(x, y) = E_n g(x, y) \tag{34}$$

where H is given by the expression (12), $g(x, y) = \phi_n(x)\exp(iny)$, and E_n determines the energy spectrum.

Equation (36) when written in full assumes the form

$$-k^2\frac{\partial^2\phi_n}{dx^2}+(u^2+2nk'u+n^2k'^2-E_n)\phi_n=0 \tag{35}$$

or

$$-k^2\frac{\partial^2\phi_n}{\partial x^2}+\mu_n\phi_n=0 \tag{36}$$

where the n-dependent function μ_n is given by

$$\mu_n=u^2(x)+2nk'(x)u(x)+n^2k'^2-E_n \tag{37}$$

In Lahiri *et al.* (1988) μ_n was erroneously written in a somewhat different form.

To solve equation (36), we recognize that when $a^2=1$, the operators A, A^+, and A_3 may be identified with the generators of a $U(1, 1)$ group. Also in this case one can solve (11) to obtain the solution (Jannussis *et al.*, 1990)

$$k(x)=x; \qquad k(x)=\frac{\sin \mu x}{\mu}; \qquad k(x)=\frac{\cos \nu x}{\nu} \tag{38}$$

Corresponding to these solutions, equation (36) reads respectively

$$x^2\frac{d^2\phi_n}{dx^2}+(E_n\mp 2nx-x^2-n^2)\phi_n=0 \tag{39}$$

$$\frac{\sin^2\mu x}{\mu^2}\frac{d^2\phi_n}{dx^2}+\left[E_n\mp\frac{2n\cos\mu x\sin\mu x}{\mu}-\frac{\sin^2\mu x}{\mu^2}-n^2\cos^2\mu x\right]\phi_n=0 \tag{40}$$

$$\frac{\cos^2\nu x}{\nu^2}\frac{d^2\phi_n}{dx^2}+\left[E_n\pm 2n\frac{\sin\nu x\cos\nu x}{\nu}-\frac{\cos^2\nu x}{\nu^2}-n^2\sin^2\nu x\right]\phi_n=0 \tag{41}$$

It may be remarked that equations (40) and (41) have the same spectrum of eigenvalues, because for $\nu x\to\nu x+\pi/2$, equation (41) is transformed into equation (40). Further, a comparison between equations (40) and (41) shows the presence of a term ± 2 (odd function of x) in them. So without restriction (Jannussis *et al.*, 1990) we may consider x to be positive.

Let us take the solution of equation (39) in the form

$$\phi_n(x)=\exp(-nx)f_n(x) \tag{42}$$

where the function $f_n(x)$ satisfies

$$x^2\frac{d^2f_n}{dx^2}-2nx^2\frac{df_n}{dx}+(E_n-n^2\mp 2nx-x^2-n^2x^2)f_n(x)=0 \tag{43}$$

Further, for $f_n(x)$ we consider a power series expansion

$$f_n(x) = \sum_{s=0} c_s x^{\rho+s} \tag{44}$$

We immediately find that the coefficients satisfy the two-term recursion equation

$$[(\rho+s)(\rho+s-1)+E_n-2n^2-1]c_s = [2n(\rho+s-1)\pm 2n]c_{s-1} \tag{45}$$

For $s=0$ we get

$$\rho(\rho-1)-1-2n^2 = -E_n$$

or

$$\rho = \tfrac{1}{2} \pm [\tfrac{1}{4} - (E_n - 1 - 2n^2)]^{1/2} \tag{46}$$

This means that for $s \to s+1$ we can write equation (45) as

$$[(\rho+s)(\rho+s+1)+E_n-1-2n^2]c_{s+1} = 2n[(\rho+s)\pm 1]c_s \tag{47}$$

From (47) for $s=S$ and $c_S \neq 0$ we get

$$\rho = -S \mp 1 \tag{48}$$

$$c_{S+m} = 0; \qquad m = 1, 2, \ldots \tag{49}$$

In consequence from (46) we have the eigenvalues

$$\begin{aligned} E_{n,S} &= -\rho(\rho-1)+1+2n^2 \\ &= -[(-S\mp 1)(-S\mp 1-1)-(1+2n^2)] \end{aligned} \tag{50}$$

The formula of the eigenvalues (30) for $n = S = 0$ gives

$$E_{00}^{+} = 1 \qquad \text{and} \qquad E_{00}^{-} = -1 \tag{51}$$

Taking all the values of n and S, we have

$$\begin{aligned} E_{nS}^{+} &= -[S(S-1)-(1+2n^2)] \\ E_{nS}^{-} &= -[(S+1)(S+2)-(1+2n^2)] \end{aligned} \tag{52}$$

which implies $E_{n,S+2}^{+} = E_{n,S}^{-}$.

6. CONCLUSION

In summary, we have considered some solvable potentials within a framework of supersymmetric quantum mechanics. The model we have explained can be described by a set of creation and annihilation operators A and A^+ which along with A_3 satisfy a four-parameter group structure. Further, we have solved an eigenvalue equation and have shown that when these operators correspond to the generators of a $U(1, 1)$ group, a two-term recursion equation (involving the energies E_n) is immediately implied.

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